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Massless radiation from strings: quantum spectrum average statistics and cusp-kink configurations

Roberto Iengo

International School for Advanced Studies (SISSA)
Via Beirut 2-4, I-34013 Trieste, Italy, and
INFN, Sezione di Trieste
E-mail: iengo@sissa.it

Abstract: We derive general formulae for computing the average spectrum for Bosonic or Fermionic massless emission from generic or particular sets of closed superstring quantum states, among the many occurring at a given large value of the number operator. In particular we look for states that can produce a Bosonic spectrum resembling the classical spectrum expected for peculiar cusp-like or kink-like classical configurations, and we perform a statistical counting of their average number. The results can be relevant in the framework of possible observations of the radiation emitted by cosmic strings.

Keywords: Long strings, String theory and cosmic strings.

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Contents

1. Introduction

In this paper we study the quantum massless radiation, both Bosonic and Fermionic, from excited closed superstrings. (For a general study of the decay of superstrings by precise numerical computations see [1 − 5]). In particular we look for string states that, in some range of the radiation energy, produce a spectrum with some (bosonic) characteristics found in the classical approximation, namely the interesting cases of classical *cusps* or *kinks* [6, 7].

The quantum spectrum is expected to agree possibly with the classical rersults in a low energy range, that is for wavelengths much larger than $\sqrt{\alpha'}$, of course.

We will try to be as general as possible and therefore we do not write in detail particular string states that give a particular spectrum. Rather, we find that a spectrum resembling, for instance, classical cusp-kink characteristics occurs on average for string configurations in which the mode excitations satisfy a kind of sum rule. We can thus further count the number of such strings satisfying that constraint. This result can be useful for evaluate how much it is likely to find that particular spectrum among the various signals possibly arriving from cosmic strings (for general studies of cosmic strings see [8, 6, 9 − 13]).

Our method is based on the observation that average radiation spectra from strings and their properties are easily derived in a suitable LightCone (LC) gauge, thus working directly with physical states and avoiding the ghost formalism.

The section 2 summarizes (and generalizes) the classical analysis.

In section 3 we introduce the convenient LC gauge and we derive the main quantum formula. It is surprisingly simple both for Bosonic and Fermionic massless emission.

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In section 4 we derive the generic (Bosonic and Fermionic) average radiation spectrum from a string of large mass, by using the quantum formula in the LC gauge and the standard statistical mechanics method of the chemical potential.

In section 5 we modify the chemical potential by introducing a suitable constraint in the average, in the form of a weight depending on a sum over the mode occupation numbers. We show that, in this way, a *cusp* or *kink* like spectrum is obtained in some sizable radiation energy domain. We then estimate the number of such string states and how rare they are among the generic set of states of a given large mass.

In the appendix A we review the (classical) relation between the gauge where ∂X^0 is constant (which we call the TP Temporal gauge) and the gauge where ∂X^+ is constant (which is called the LC LightCone gauge). That relation can also be seen as an algorithm for obtaining (classical) solutions in the TP gauge which will automatically satisfy the Virasoro constraints.

In the appendix B we construct a sample of a generic classical string state in the Temporal gauge, following the results of section 4 and the recipes of appendix A (for a recent portait of the string based on the computation of form factors see [13]).

In the appendix C we discuss the particular case of the state of maximal angular momentum, which classically is an example of a cusp. We compute and compare the classical and the quantum spectrum for the mass $M = \sqrt{4N/\alpha'}$ with $N = 1000$. Even for this large value of N, there is no radiation energy region for which the quantum spectrum matches the classical cusp behaviour. The classical and quantum spectra show a good agreement for small radiation energy, where most of the radiation occurs, but where however the classical behaviour is not yet of the cusp form; for larger radiation energies, where the classical spectrum is cusp-like, the quantum spectrum falls off to zero much more rapidly than the classical one.

2. The classical computation

We begin by reviewing the classical massless radiation rate of a closed string (see [7, 4]):

$$rate = g_s^2 \frac{p_0^{D-3}}{M^2} \int d\Omega \sum_{\xi,\tilde{\xi}} |\xi_j I_R^j \tilde{\xi}_k I_L^k|^2 \,. \tag{2.1}$$

Here D is the number of extended dimensions, $M = \sqrt{4N/\alpha'}$ is the mass of the string (assumed at rest), p_0 is the energy of the emitted massless particle (graviton, scalar or antisimmetric tensor), $\xi_j \tilde{\xi}_k$ its polarization and

$$I_{R,L}^j = \int d\sigma_\pm e^{ip_\mu X_{L,R}^\mu(\sigma_\pm)} \partial X_{R,L}^j(\sigma_\pm) \,. \tag{2.2}$$

This computation is usually done in the *Temporal gauge* where $X_{L,R}^0 = \alpha'(M/2)\sigma_\pm$ with $\sigma_\pm = \tau \pm \sigma$ such that $X^0 = X_L^0 + X_R^0 = \alpha' M \tau$.

There is a saddle point [7] in the integral defining $\xi \cdot I_{L,R}$ if

$$p \cdot \partial X_{L,R} = 0 \; at \; some \; \sigma_\pm = \sigma_\pm^c \,. \tag{2.3}$$

We can take $\sigma_\pm^c = 0$. This is the condition defining a *cusp*.

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Another interesting case is when $p \cdot \partial X_{L,R}$ has a discontinuity. This is referred as the *kink*.

To be precise, a *cusp* or a *kink* occur when the above conditions are respectively satisfied in both the Left and the Right sectors.

However, since our study can be done separately and independently for each sector, *from now on we discuss, say, the Left sector only, and write* X *meaning* X_L *and* $\xi \cdot I$ *meaning* $\xi \cdot I_L$ *and* σ *meaning* σ_+. Of course the other (R) sector is treated in the same way.

Take the frame where $p_\mu = (p_0, p_z, 0)$ with $p_z = -p_0$. In this frame $p_+ = \frac{p_0+p_z}{\sqrt{2}} = 0$ and the cusp condition is $\partial X^+ = 0$.

In the "temporal gauge" the *cusp* is only possible if also $\partial X^T = 0$ (for every T-ransverse component). In fact in the Temporal gauge $\partial X^+ + \partial X^- =$ *constant* and it follows from the classical Virasoro constraints $2\partial X^+ \partial X^- = (\partial X^T)^2$ that if $\partial X^T \to 0$ then $\partial X^+ \sim (\partial X^T)^2$.

Assuming that ∂X^T vanishes linearly we have

$$p_- X^+ \sim \sigma^3 \,. \tag{2.4}$$

In this case, for large $N_0 \equiv \sqrt{\alpha' N} p_0$ we can extend the integration over σ to $-\infty, +\infty$ and we get

$$\xi \cdot I = \int d\sigma e^{ip_- X^+} \xi_T \cdot \partial X^T \sim \int d\sigma \sigma e^{icN_0 \sigma^3} \sim N_0^{-2/3} \,. \tag{2.5}$$

In general it could be that $\partial X^T \sim \sigma^\beta$ and thus $p_- X^+ \sim \sigma^{2\beta+1}$. In this case

$$\xi \cdot I \sim N_0^{-\frac{\gamma}{2}} \quad with \quad 1 < \gamma = \frac{2\beta+2}{2\beta+1} \leq 2 \,. \tag{2.6}$$

This general behavior in N_0 includes the result for the *kink* for which $\beta = 0 \to \gamma = 2$. *We will refer to all these cases with the various possible* γ *as "cusp".*

3. The quantum computation

The quantum expression of the rate is the same as eq. (2.1) with $\xi \cdot I$ given by the relevant quantum matrix element [4]. The quantum computation is most easily done in a Light-Cone (LC) gauge, where the Fock space of the T-ransverse oscillators comprises all the physical states. We specify the LC gauge by taking LC coordinates such that $p_+ = 0$ as above.

In the LC-gauge $X^+ = \alpha'(M/2)\sigma$ (remember that we mean $X_L^+ = \alpha'(M/2)(\sigma + \tau)$). *Also the classical computation for the cusp can be performed in the LC-gauge.*

This amounts simlpy to a change of integration variable in eq. (2.4)

$$\sigma \to \sigma' = \sigma^3 \,. \tag{3.1}$$

In this gauge, that is in this new variable, ∂X^T is divergent $\sim \sigma'^{-1/3}$ (or $\sim \sigma'^{-\beta/(2\beta+1)}$ in the general case), rather than going to zero (see also appendix A).

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Quantum computation. In the $p_+ = 0$ LC gauge the (Left or Right) part of the vertex operator for emitting a massless NSNS state is (we can take it at $\sigma = 0$)

$$V(0) = \xi_T \cdot \partial X^T(0) e^{ip_- X^+(0)} = \xi_T \cdot \partial X^T(0) \,. \tag{3.2}$$

Thus we have

$$|\xi \cdot I|^2 = \sum_f |\langle f | \xi_T \cdot \partial X^T | \Phi_N \rangle|^2 \,, \tag{3.3}$$

where $|\Phi_N >$ represents the radiating state with mass $M = 2\sqrt{N/\alpha'}$, which is supposed to be at rest, and $|f>$ is a possible final state with mass $M' = 2\sqrt{N'/\alpha'}$. Let us take here $\alpha' = 4$. The radiated energy is $p_0 = (M^2 - M'^2)/2M = N_0/2\sqrt{N}$ with $N_0 = N - N'$.

The vertex is linear in the transverse oscillator operators and the computation is easy.

We are interested in the average (in particular for $N \to \infty$) over the many different states $|\Phi_N >$ which share some properties.

For a definite $p_0 = N_0/2\sqrt{N}$, that is for a definite N_0, we have

$$\langle |\xi \cdot I_L(p_0)|^2 \rangle = \frac{1}{\mathcal{N}} Tr[(\xi_T \cdot \partial X^T)^\dagger_{N_0} \; (\xi_T \cdot \partial X^T)_{N_0}]_N \,. \tag{3.4}$$

The trace is restricted to initial states with fixed N ($\mathcal{N}$ being their number) and moreover $(\xi_T \cdot \partial X^T)_{N_0}$ means restricting the operator to that part that lowers the value of the number operator $\hat{N}$ from N to the final state value $N' = N - N_0$.

In terms of the transverse oscillators

$$(\xi_T \cdot \partial X^T)_{N_0} = \sqrt{\frac{\alpha'}{2}} N_0^{1/2} \xi \cdot a_{N_0} \,. \tag{3.5}$$

The normalization is $[a^i_{+n}, a^j_{-m}] = \delta^{ij}\delta_{nm}$ and we take $\alpha' = 4$.

Therefore, $p_0 = N_0/2\sqrt{N}$ being the radiated energy, we get THE MAIN FORMULA

$$\sum_\xi \langle |\xi \cdot I(p_0)|^2 \rangle = \sum_\xi \left\langle [V^\dagger V]_{N_0} \right\rangle = \sum_\xi \langle |\xi \cdot \partial X|^2_{N_0} \rangle = 2 \langle N_0 a_{-N_0} \cdot a_{N_0} \rangle \,. \tag{3.6}$$

The above formula describes the NS radiation (say, in the Left sector). In the LC one easily get also the corresponding formula for the R(amond) radiation by using the Green-Schwarz formalism (see [14]).

We remember that in this formalism the fermionic degrees of freedom are carried by the S^a_n oscillator operators, $a = 1, \cdots, 8$ being a spinor index, satisfying $\{S^a_{+n}, S^b_{-m}\} = \delta^{ab}\delta_{nm}$.

Since the emitted momentum satisfies $\vec{p}_T = p_+ = 0$, the vertex for emitting the (Left part of) a massless fermion is $V_F(0) = u \cdot S(0)\sqrt{\hat{P}_+}$ where $u(p)$ is a suitably normalized polarization spinor and $\hat{P}_\mu$ is the momentum operator.

By averaging over u (we take $\sum_u u^\dagger_a u_b = 2\sqrt{\alpha'} p_0 \delta_{a,b}$) we find

$$\sum_u \langle |u \cdot I_F(p_0)|^2 \rangle = \sum_u \left\langle [V_F^\dagger V_F]^2_{N_0} \right\rangle = 2 \langle N_0 S_{-N_0} \cdot S_{+N_0} \rangle \tag{3.7}$$

We remember that in terms of the LC oscillators the number operator is $\hat{N} = \sum_{n>0} (n a_{-n} \cdot a_{+n} + n S_{-n} \cdot S_n)$.

4. The average spectrum

We will call $\langle 2na_{-n}\cdot a_n\rangle$ *the spectrum for the Bosonic radiation, or* $\langle 2nS_{-n}\cdot S_{+n}\rangle$ *for the Fermionic one, although this is only the Left part, and to obtain the physical spectrum one has to take the product of Left and Right times the phase space* $\frac{p_0^{D-3}}{M^2}\Omega$.

Now we review the derivation of the general average spectrum that is taking the average over all the states with $\left\langle \hat{N}\right\rangle = N$ [15], [4].

Mimicking statistical mechanics introduce a chemical potential term $e^{-\hat{N}\epsilon}$ and, beginning with the Bosonic spectrum, replace

$$Tr[na_{-n}\cdot a_n]_N \rightarrow Tr[na_{-n}\cdot a_n e^{-\hat{N}\epsilon}] \tag{4.1}$$

and we fix ϵ requiring $\left\langle \hat{N}\right\rangle = N$. We get

$$\begin{aligned}\langle na_{-n}\cdot a_n\rangle + \langle nS_{-n}\cdot S_n\rangle &= \frac{1}{\mathcal{N}}Tr[(na_{-n}\cdot a_n + nS_{-n}\cdot S_n)e^{-\hat{N}\epsilon}] \\ &= D_T\; n\left\{\frac{e^{-n\epsilon}}{1-e^{-n\epsilon}} + \frac{e^{-n\epsilon}}{1+e^{-n\epsilon}}\right\}\end{aligned} \tag{4.2}$$

where D_T is the number of tranverse dimensions.

Similarly

$$\mathcal{N} = Tr[e^{-\hat{N}\epsilon}] = \prod_n \left(\frac{1+e^{-n\epsilon}}{1-e^{-n\epsilon}}\right)^{D_T} \tag{4.3}$$

ϵ is fixed by requiring

$$N = D_T\sum_n\left\{\frac{ne^{-n\epsilon}}{1-e^{-n\epsilon}} + \frac{ne^{-n\epsilon}}{1+e^{-n\epsilon}}\right\} = -\frac{d}{d\epsilon}D_T\sum_n \log\left[\frac{1+e^{-n\epsilon}}{1-e^{-n\epsilon}}\right] \tag{4.4}$$

(we recognize the standard saddle-point equation of string theory).

For small ϵ

$$D_T\sum_n \log\left[\frac{1+e^{-n\epsilon}}{1-e^{-n\epsilon}}\right] \rightarrow \frac{D_T}{\epsilon}(c_F + c_B) \tag{4.5}$$

with $c_F = \int dx \log[1+e^{-x}] = \pi^2/12$ and $c_B = -\int dx\log[1-e^{-x}] = \pi^2/6$.

Therefore $\epsilon = \sqrt{D_T(c_F+c_B)/N}$ and we get for large N

$$\begin{aligned}\log[\mathcal{N}] &\sim 2\sqrt{ND_T(c_F+c_B)} \\ \sum_\xi |\xi\cdot I(p_0)|^2 &= 2\frac{N_0 e^{-N_0\sqrt{D_T(c_F+c_B)/N}}}{1-e^{-N_0\sqrt{D_T(c_F+c_B)/N}}}\end{aligned} \tag{4.6}$$

(remember $N_0 = \sqrt{\alpha' N}p_0$). Thus we get a thermal-like ($Left$ part of the) spectrum with a temperature $\sim 1/\sqrt{\alpha'}$.

By repeating the computation for the ($Left$ part of the) Fermionic spectrum it easily seen that one gets a Fermi-Dirac distribution

$$\sum_u |u\cdot I_F(p_0)|^2 \sim 2\frac{N_0 e^{-N_0\sqrt{D_T(c_F+c_B)/N}}}{1+e^{-N_0\sqrt{D_T(c_F+c_B)/N}}}\,. \tag{4.7}$$

5. The average quantum cusp-kink-like spectrum

Looking for the quantum states corresponding to the classical cusp or kink. In this case we look for the Bosonic radiation.

The classical *cusp* expression for $\xi \cdot I$ eq. (2.6) is obtained for a particular angle of the radiation momentum $\vec{p}$, namely the one for which $\partial X^+(\sigma) = 0$ at some σ, where $X^+ = (p_0 X^0 - \vec{p}\vec{X})/p_0$. Taking the $p_+ = 0$ Light-Cone frame and looking for a quantum spectrum corresponding to the classical *cusp* one, we implicitly select some particular direction for the polarization of the quantum states relative to the direction of the emitted momentum.

By putting in eq. (3.6) the classical *cusp* expression for $\xi \cdot I$ eq. (2.6) one expects that, for the states corresponding to the classical *cusp*, it holds $n^{\gamma+1} \langle a_{-n} a_n \rangle = A$ with A constant, strictly speaking for $n \gg 1$ (remember that we are considering just the Left component).

For the classical *cusp* $\sum_1^{n^c} n^{\gamma+1} a_{-n} a_n = A \cdot n^c$ divergent for $n^c \to \infty$.

However, the classical behaviour can only hold up $n \leq n^c \ll N^{1/2}$, that is when the radiated energy $p_0 = n/\sqrt{\alpha' N}$ is much less than the inverse of the string length $1/\sqrt{\alpha'}$.

Thus we assume $n^c \sim N^\alpha$ with $0 < \alpha < 1/2$ and *we take as a definition of quantum cusp states the requirement* $\sum_1^{n^c} n^\gamma |\xi \cdot I(n)|^2 = \sum_1^{n^c} n^{\gamma+1} < a_{-n} a_n >= A \cdot n^c$.

As for the value of A: in the literature [6, 7] it is assumed that for a generic cusp or kink A is of the order of N. [1] We keep this assumption to see its implications. Actually we will find that $A \sim N$ corresponds to quite rare, rather than generic, configurations.

There is a constraint on A since it must be $\sum_1^{n^c} < n a_{-n} a_n >= qN$ with $q < 1$. For large n_c this means that $A = q/k(\gamma) N$ (for the standard $\gamma = 4/3$ cusp $k(\gamma) \sim 3.6$). We take q as a parameter; a very small q corresponds to a rather irrelevant cusp.

In the interval $\mathcal{I} = \{1 \ll n < n^c\}$ we deform the chemical potential

$$e^{-n a_{-n} a_n \epsilon} \to e^{-n^{\gamma+1} a_{-n} a_n \eta} \tag{5.1}$$

while keeping $e^{-n a_{-n} a_n \epsilon}$ for $n > n_c$. The regions $n = O(1)$ and $n = O(n_c)$ are left unspecified as they do not play an important role in the following.

We get the spectrum

$$\langle n a_{-n} a_n \rangle = D_T \frac{n e^{-n^{\gamma+1}\eta}}{1 - e^{-n^{\gamma+1}\eta}} \quad for\ n \subset \mathcal{I} \tag{5.2}$$

$$\langle n a_{-n} a_n \rangle = D_T \frac{n e^{-n\epsilon}}{1 - e^{-n\epsilon}} \qquad for\ n > n^c \tag{5.3}$$

1) Fix η requiring $\sum_1^{n^c} \langle n^{\gamma+1} a_{-n} a_n \rangle = A \cdot n^c$.
   By taking $n_c \leq N^{\frac{1}{\gamma+1}}$ we have the solution $\eta \sim D_T A^{-1}$ since in this case for $n \subset \mathcal{I}$ we recover the *cusp spectrum*

$$\langle n a_{-n} a_n \rangle = D_T \frac{n e^{-n^{\gamma+1}\eta}}{1 - e^{-n^{\gamma+1}\eta}} \to A \frac{1}{n^\gamma}\,. \tag{5.4}$$

[1] when comparing, remember that our convention for the temporal gauge is $X^0 = \alpha' M \tau$ whereas in the literature on cosmic strings it is often $X^0 = \tau$; in the latter convention $A \sim N$ corresponds to $\partial^2 X_T \sim 1/M$
.

2) In order to fix ϵ we require

$$(1-q)N = D_T \left\{ \sum_{n^c}^{\infty} \frac{n e^{-n\epsilon}}{1-e^{-n\epsilon}} + \sum_{1}^{\infty} \frac{n e^{-n\epsilon}}{1+e^{-n\epsilon}} \right\}. \tag{5.5}$$

For large n we approximate the sum with the integral, like in the previous section. We have

$$\begin{aligned} \sum_{n^c}^{\infty} \frac{n e^{-n\epsilon}}{1-e^{-n\epsilon}} &\to \frac{1}{\epsilon^2} \int_{n_c\epsilon}^{\infty} dx x \frac{d}{dx} \log[1-e^{-x}] \\ &= \frac{1}{\epsilon^2} \left(- \int_{n_c\epsilon}^{\infty} dx \log[1-e^{-x}] - n_c\epsilon \log[1-e^{-n_c\epsilon}] \right) \\ &\approx -\frac{1}{\epsilon^2} \int_0^{\infty} dx \log[1-e^{-x}] = \frac{c_B}{\epsilon^2}. \end{aligned} \tag{5.6}$$

The pre-last step holds for $n_c\epsilon \to 0$. In fact, for N large we get $(1-q)N = \frac{c_B+c_F}{\epsilon^2}$ and thus $\epsilon = N^{-1/2}\sqrt{D_T(c_F+c_B)/(1-q)}$.

3) The number of these *cusp* states is

$$\log[\mathcal{N}_\gamma] = D_T \left\{ -\sum_{1}^{n^c} \log[1-e^{-n^{\gamma+1}\eta}] - \sum_{n^c}^{\infty} \log[1-e^{-n\epsilon}] + \sum_{1}^{\infty} \log[1+e^{-n\epsilon}] \right\}. \tag{5.7}$$

Note that

$$-\sum_{n^c}^{\infty} \log[1-e^{-n\epsilon}] + \sum_{1}^{\infty} \log[1+e^{-n\epsilon}] \approx \frac{1}{\epsilon}(c_B + c_F + n_c\epsilon \log[1-e^{-n_c\epsilon}]) \tag{5.8}$$

4) In conclusion we get

$$\log[\mathcal{N}_\gamma] = a \cdot \bar{n} + 2\sqrt{(1-q)D_T(c_F+c_B)} \cdot N^{\frac{1}{2}} - n_c \log\left[N^{\frac{1}{2}}/n_c\right] \tag{5.9}$$

where a is some constant and $\bar{n}$ is the minimum between n_c and $N^{\frac{1}{\gamma+1}}$.

For instance in the case $\gamma = 4/3$ and $n_c \sim N^{\frac{1}{\gamma+1}}$ we get for the log of the ratio of the cusp number to the general average number

$$\log\left[\frac{\mathcal{N}_{4/3}}{\mathcal{N}}\right] = -2\sqrt{D_T(c_F+c_B)} \cdot \left(1-\sqrt{1-q}\right) \cdot N^{1/2} + (a - b\log[N]) \cdot N^{3/7}. \tag{5.10}$$

Therefore those cusps are very rare within the variety of the generic string states.

That fact could have been already guessed by observing how different is the N dependence of the spectrum for $1 \ll n < n_c$: it is $\langle n a_{-n} a_n \rangle \sim N/n^\gamma$ in the *cusp* configurations whereas $\langle n a_{-n} a_n \rangle \sim N^{1/2}$ for the generic string state.

In the generic case the dominant contribution to the sum rule $\sum_n \langle n a_{-n} a_n \rangle = N$ comes from $n \sim N^{1/2}$, whereas in the *cusp* configurations the region $n \ll N^{1/2}$ gives a substantial fraction of the result.

Taking a higher value for n_c, say $n^c \sim N^{1/2}$, would give an even smaller fraction since the main difference with the above computation would be replacing c_B with $\tilde{c}_B < c_B$ in the expression of $\log[\mathcal{N}_\gamma]$.

In order to find more abundant cusp configurations we should assume $A \sim N^{1/2}$. In this case $q \sim N^{-1/2}$ in (5.9): we find more states but the possibly observed radiation would be more feeble and therefore the *cusp-like* characterization of the signal becomes rather marginal.

A. Temporal and lightcone gauges

We consider a string at rest with four-momentum $P_\mu = (M, \vec{0})$. Here we put $\alpha' = 1$.

The Virasoro constraints are, for the Left or Right part,

$$(\partial X^0_{L,R})^2 = (\partial Z_{L,R})^2 + (\partial \vec{X}_{L,R})^2 \tag{A.1}$$

where ∂ is the derivative with respect to the World-Sheet (WS) parameter $s_{L,R} = \tau \pm \sigma$ which is different for different gauge choices. We have chosen a $Z-$direction thus $\vec{X}$ is defined to be transverse. Let us consider for instance the Left part (dropping the suffix "L").

In the *Temporal (TP) gauge* one takes $\partial_{\hat{s}} X^0_{TP} = \frac{M}{2}$ where we call $\hat{s}$ the WS parameter.

In the *LightCone (LC) gauge* one takes $\partial_s X^+_{LC} \equiv \partial_s X^0_{LC} + \partial_s Z_{LC} = \frac{M}{2}$ where we call s the new WS parameter.

Classically, the passage between TP and LC is a redefinition of the WS parameter $\hat{s} \to s$, that is $\vec{X}_{LC}(s) = \vec{X}_{TP}(\hat{s}(s))$ and similarly for X^0, Z:

$$(\partial_{\hat{s}} X^0_{TP} + \partial_{\hat{s}} Z_{TP})\frac{\partial \hat{s}}{\partial s} = \frac{M}{2} \quad \Rightarrow \quad \frac{\partial \hat{s}}{\partial s} = \frac{1}{1 + \frac{2\partial_{\hat{s}} Z_{TP}}{M}} \tag{A.2}$$

Note that, because of the constraint, $|\partial_{\hat{s}} Z_{TP}| \leq M/2$ and therefore both $\hat{s}(s)$ and $s(\hat{s})$ are well defined. It follows that

$$\partial_s \vec{X}_{LC} = \frac{\partial_{\hat{s}} \vec{X}_{TP}}{1 + \frac{2\partial_{\hat{s}} Z_{TP}}{M}} \tag{A.3}$$

For instance, we see that even if $\partial_{\hat{s}} \vec{X}_{TP}$, $\partial_{\hat{s}} Z_{TP}$ only contain one Fourier mode of the WS parameter in the TP (like it is for the maximal angular momentum string configuration), $\partial_s \vec{X}_{LC}$ will in general contain all the Fourier modes of the WS parameter in the LC.

Viceversa in the LC gauge we have

$$\partial_s X^-_{LC} = \frac{2|\partial_s \vec{X}_{LC}|^2}{M} \quad \Rightarrow \quad \partial_s X^0_{LC} = \frac{M}{4} + \frac{|\partial_s \vec{X}_{LC}|^2}{M} \tag{A.4}$$

$$\frac{\partial s}{\partial \hat{s}} \partial_s X^0_{LC} = \frac{M}{2} \quad \Rightarrow \quad \frac{\partial s}{\partial \hat{s}} = \frac{2}{1 + \frac{4|\partial_s \vec{X}_{LC}|^2}{M^2}} \tag{A.5}$$

and therefore

$$\partial_{\hat{s}} \vec{X}_{TP} = \frac{2\partial_s \vec{X}_{LC}}{1 + \frac{4|\partial_s \vec{X}_{LC}|^2}{M^2}}\,. \tag{A.6}$$

It can be checked that (A.2) and (A.6) are consistent.

We see that the classical relation between TP and LC is highly nonlinear. The quantum version of it is to our knowledge not available.

B. The shape of a generic string

From the form of the (bosonic) spectrum eq. (4.6) and the main formula eq. (3.6) we can reconstruct the corresponding classical transverse string in the LC gauge (consider here the Left component), by putting for the transverse part:

$$X^i_{LC}(s) = \sum_n c^i_n Cos\left[ns + \theta^i_n\right] \tag{B.1}$$

where θ^i_n are random phase shifts, in general different for Left and Right, and

$$\sum_i (c^i_n)^2 = A\frac{1}{n}\frac{e^{-\frac{n}{g\sqrt{N}}}}{1-e^{-\frac{n}{g\sqrt{N}}}}\,. \tag{B.2}$$

Here $g = 1/\sqrt{D_T(c_B + c_F)}$ and $A = \sum A^i$ where the values of A^i are randomly distributed among the transverse directions.

For the sake of simplicity we will consider the particular example where only one of the X^i_{LC} is different from zero. Therefore our sample string is less random than the true generic one. We conventionally take $A = g = 1$.

We do the same for the Right component, with different random phase shifts.

We have taken $N = 100$ (and cutoff the sum at $n = 100$). We assume that the string state is at rest and its mass is, according to the LC formulae,

$$M = P_+ = P_- = 2\sqrt{\frac{\int_0^{2\pi} ds(\partial_s X_{LC}(s))^2}{2\pi}}\,. \tag{B.3}$$

The result is the same for Left and Right as it should be, and M is proportional to $\sqrt{N}$.

According to the LC prescription we put (Left component)

$$X^+_{LC} = \frac{M}{2}s \quad X^-_{LC} = \frac{M}{2}s + \left\{\frac{\int_0^s ds'(\partial_s X_{LC}(s'))^2}{2\pi} - \frac{M}{2}s\right\}. \tag{B.4}$$

The part of X^- which is in curly brackets is taken to be 2π-periodic, that is, its value at $s + 2\pi$ is identified with its value at s. Further, from the previous formulae,

$$Z_{LC} = -\left\{\frac{\int_0^s ds'(\partial_s X_T(s'))^2}{2\pi} - \frac{M}{2}s\right\}. \tag{B.5}$$

In order to get the string in the Temporal gauge we use eq. (A.5) to get $s(\hat{s})$ and take $X_{TP}(\hat{s}) = X_{LC}(s(\hat{s}))$, $Z_{TP}(\hat{s}) = Z_{LC}(s(\hat{s}))$.

We do the same for the Right part and finally we get in the TP gauge

$$\begin{aligned} X_{TP}(\tau,\sigma) &= X_{TP}(Left)(\tau-\sigma) + X_{TP}(Rigth)(\tau+\sigma) \\ Z_{TP}(\tau,\sigma) &= Z(_{TP}Left)(\tau-\sigma) + Z_{TP}(Rigth)(\tau+\sigma) \end{aligned} \tag{B.6}$$

In figures 1–3) we show the resulting (TP) string in the plane X, Z for some values of $\tau = 0, \pi/4, \pi/2$.

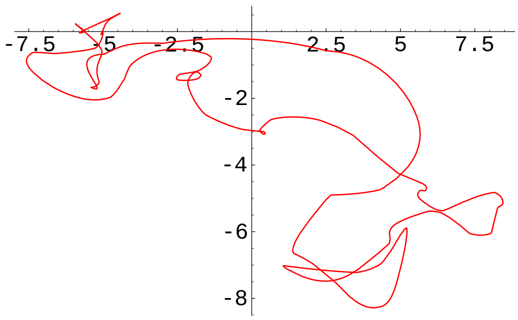

Figure 1: A generic string at $\tau = 0$.

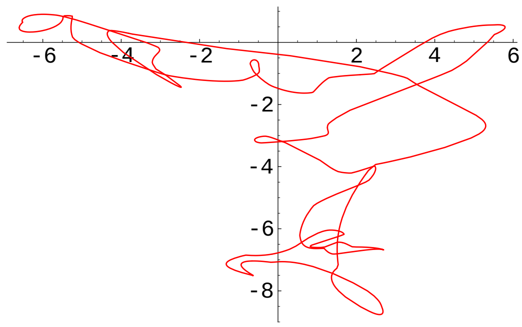

Figure 2: A generic string at $\tau = \pi/4$.

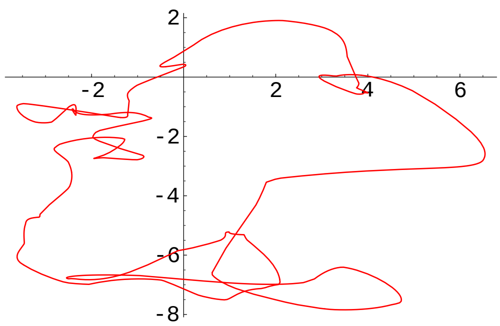

Figure 3: A generic string at $\tau = \pi/2$.

C. A particular cusp-like case: the maximal angular momentum state

The classical state of a closed string of maximal angular momentum is, in the Temporal Gauge,

$$\frac{X_L^1 + iX_L^2}{\sqrt{2}} = \frac{L}{2\sqrt{2}} e^{i(\tau+\sigma)} \qquad \frac{X_R^1 + iX_R^2}{\sqrt{2}} = \frac{L}{2\sqrt{2}} e^{i(\tau-\sigma)} \tag{C.1}$$

that is

$$\frac{X^1 + iX^2}{\sqrt{2}} = \frac{X_L^1 + iX_L^2}{\sqrt{2}} + \frac{X_R^1 + iX_R^2}{\sqrt{2}} = \frac{L}{\sqrt{2}} e^{i\tau} cos(\sigma) \tag{C.2}$$

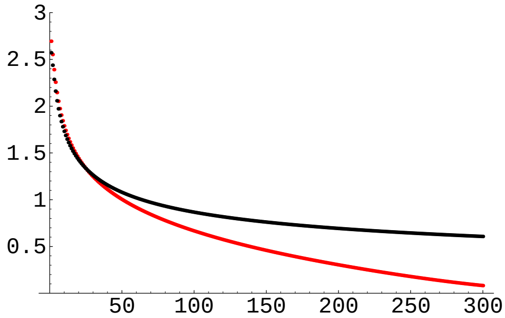

Figure 4: Classical (black) and quantum (red) spectrum times $n^{4/3}$.

and $X^0 = \alpha' M\tau = L\tau$. In the quantum state $M = 2\sqrt{N/\alpha'}$ where the integer N is the eigenvalue of the number operator $\hat{N}$.

One would think to represent the corresponding quantum state by a coherent state of the oscillators. However due to the Virasoro constraints this coherent state would not have a definite mass and therefore X^0 would be undefined. Luckily we do not need that, since we know precisely the unique quantum maximal angular momentum state in the Temporal Gauge:

$$|\Psi^{Jmax} >= \frac{(b_{-1})^N \psi^b_{-1/2}[0>_L}{\sqrt{N!}} \otimes \frac{(\tilde{b}_{-1})^N \tilde{\psi}^b_{-1/2}[0>_R}{\sqrt{N!}} \tag{C.3}$$

where $b_{-1} = \frac{a^1_{-1}+ia^2_{-1}}{\sqrt{2}}$, $\psi^b_{-1/2} = \frac{\psi^1_{-1/2}+i\psi^2_{-1/2}}{\sqrt{2}}$. Therefore we can compute $\sum_\xi |\xi \cdot I_L|^2$ both classically and quantum mechanically.

Classically one has radiation of the bosonic part of the graviton multiplet therefore we compare with the quantum NS massless emission.

The relevant formulae are written in [4], that is, referring to [4], the modulus square of eq. (3.45) for the classical radiation and eqs. (3.7), (3.8) for the quantum computation, together with the explicit expressions in (3.10)–(3.13) and in appendix B for the rest. It is important to keep all the terms of the quantum computation, which has been checked by comparing the result for $NS_L \times NS_R$ with the independent computation made by taking the imaginary part of the torus diagram and restricting the spinstructure to NS-NS.

To look for the *cusp* spectrum we take the emitted momentum to lay in the X^1, X^2 plane. The classical result for $\sum_\xi |\xi \cdot I_L|^2$ is expressed as a constant times N times a function depending on n only (remember the emitted energy $\omega = n/(2\sqrt{N})$). It reaches rather slowly the expected behavior $n^{-4/3}$ for large n.

The quantum result is a more complicated non factorized expression in terms of N and n and we have computed it for $N = 1000$ and $n \leq 300$. It is very near to the classical result for $n < 50$ (this is also a check of the computation since the normalization is fixed), where however the classical result has not yet reached the *cusp-like* asymptotic behaviour, after which it goes to zero more rapidly.

The comparison is shown in figure 4 where we show the classical (black) and quantum (red) results as a function of n, both multiplied by $c \times n^{4/3}$, choosing c such as to get the classical curve $= 1$ for $n = 1000$.

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Therefore this particular state does not seem to follow the *cusp-like* pattern and therefore its behaviour cannot be compared with the average *cusp* one. However it agrees with the generic expectation that the most important part of the radiation is emitted for low n where it matches the classical pattern. In general, that region of small n is not likely to be part of the asymptotic, possibly *cusp-like*, behaviour. If this is true, then the *cusp* characterization of the string states would not be so relevant for observations.

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